Geometry/Topology Comp Solutions

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Contents

1 Question 76.2 Munkres

This is a great question on Covering space/Deck Transformations

1.1 a

This part is straightforward: This is an index 2 covering, an all subgroups of index 2 are normal.

1.2 b

This space has deck transformation group trivial, and is not normal. Normal coverings are those in which there is a deck transformation which takes any lift of a certain point to another lift. But if we pick a point which is the lift of the center of X, it will lift to a point which has a loop A_3 attached, no other point in this covering space has such a point attached to it, so we cannot hope to send it to another point. It must therefore be trivial. Moreover $Aut(Y/X) = N(H)/H$ where H is the pushforward of the fundamental group of the cover. Since $3 = |G : H| \implies N(H)/H|3$ so it's order 3 or 1. The above discussion states that it cannot hope to be 3, it's 1 and hence is non normal

1.3 c

This is order 4 covering, so $|N(H)/H|$ is 1,2 or 4. There is a nontrivial deck transformation which flips the points at the loops b around, but it's not the whole group, this is all we can do. The deck group is $\mathbb{Z}/2\mathbb{Z}$, but it's not normal, since $N(H) \neq G$

1.4 d

We may slide this right or left integer amounts, it's a normal covering with deck group $\mathbb{Z}$.

2 August 2021

2.1 G2

a)

If

$$\int_M H^2 - K\, dA = 0$$

then we know that $H^2 - K = (\frac{k_1 - k_2}{2})^2 = 0$ thus $k_1 = k_2$ so this surface is all umbilic. A textbook theorem gives that this is part of a sphere or a plane.

2.2 G5

Let M be an orientable geometric surface with boundary, and Gaussian curvature strictly larger than 1. Let the boundary be formed by smooth geodesic curves.

a)

By Gauss Bonnet we have

$$\int_M K\,dM + \int_{\partial M} k_g\,ds = 2\pi\chi(M)$$

As the boundary is formed by smooth geodesic curves, the second term is zero. Since $K > 1$ and K is continuous the integral $\int_M K\,dM > 1$ as well. Hence we know that $2\pi\chi(M) > 1$ so the Euler characteristic must be positive.

b)

As $2\pi\chi(M) > 1$ we know that $\chi(M) > 1$, so our only choice is $\chi(M) = 2$

c)

2.3 T2

We use SvK on the space 3 times. The idea is that since the points are identified we know that

2.4 T3

Recall that the plane $\mathbb{R}^2$ is homeomorphic to the two-sphere minus a point via sterographic projection. Moreover gluing a cylinder to S^2 yields a space homeomorphic to T^2, and thus the surface in question is homeomorphic to the torus minus a point. This latter space is homotopy equivalent to $S^1 \vee S^1$, so $M \simeq S^1 \vee S^1$. The deRham cohomology groups of this space are well known.

2.5 T5

a

First we inspect the degree of these coverings. The degree is the number of preimages of points, and since both Y, Z are path connected the degrees are the same for any point in the spaces. If we look at the base points both Y, Z have 3 preimages, hence the degree of the coverings is 3 for both spaces. Recall the degree is also defined as the index of the fundamental groups of Y, Z under the induced homomorphism of the covering space. By definition $Aut(Y) = N(H)/H$ where $H = p_*(\pi_1(Y, q_2))$, we can replace Z for Y and q_2 with any preimage of q. Since the index of H is 3 we know that the normalizer being a larger subgroup than H must be either H itself or $\pi_1(X,q)$, this follows from Lagrange's theorem: $H < N(H) < \pi_1(X,q) = G$

$$3 = |G/H| = |G/N(H)||N(H)/H|$$

So $|N(H)/H| = 1$ or 3. If 1 then $N(H) = H$ so the automorphism group is trivial, if 3 then $N(H) = G$ and in particular $|Aut(Y)| = |N(H)/H| = 3 \implies Aut(Y/X) \cong \mathbb{Z}_3$

Now for our Y, Z spaces $Aut(Y) \cong 1$ and $Aut(Z) \cong \mathbb{Z}_3$. The reason being for any point in Y, say q_3, we cannot map it to any other point in Y under automorphism, no other point in Y has a b_3 loop attached to it, the other two base points q_1, q_2 have b_1, b_2 coming either in or out but not both so in the picture there is no loop, just a lift "upwards". Another way we could see this it to try to imagine rotating or reflecting this picture, there are no other possible symmetries attached to Y other than the trivial one. As for Z this has all possible symmetries, we can effectively rotate this space 120° after some manipulations of 'pulling' q_1, q_3 downwards to get a more symmetric looking graph. Another way is to just say that each of q_1, q_2, q_3 has a loop of 'a' and of 'b' (rather the preimages of a, b) coming in and out.

b

Y is not a regular covering space by the above argument, but Z is regular.

c

Y is path connected so the degrees (indexes) are all the same, and it's 3.

3 January 2021

3.1 G2

A helpful trick whenever one sees an integral with H^2 is to subtract the Gaussian curvature, this is exactly what we need for this problem:

$$\int\int H^2 - KdM = \int\int \left(\frac{k_1 - k_2}{2}\right)^2 dM \geq 0$$

Since we're integrating a nonnegative function. Thus we know $\int\int H^2 - K \geq 0$. As such we can then write

$$\begin{aligned}\int\int H^2 - KdM &= \int\int H^2 dM - \int\int KdM \\ &= \int\int H^2 dM - 2\pi\chi(M)\end{aligned}$$

Where the second line comes from Gauss Bonnet. As such this is nonnegative so we can write

$$\int\int H^2 dM \geq 4\pi$$

Since M homeomorphic to a sphere, the Euler Characteristic is 2. Now the equality will come from a textbook theorem in Chapter 6 of O'Neill which says any compact all umbilic surfaces are spheres. If we have equality then the first equation we have

$$\begin{aligned}\int\int H^2 dM = 4\pi &\iff \int\int H^2 dM = \int\int KdM \\ &\iff \int\int H^2 - KdM = 0 \\ &\iff \int\int \left(\frac{k_1 - k_2}{2}\right)^2 dM = 0\end{aligned}$$

Where the last equality means that $k_1 = k_2$, and therefore M is a compact surface which is all umbilic.

3.2 G3

Let α be an asymptotic curve on a surface M with nonzero curvature. As α is asymptotic then $k(\alpha') = k(T) = S_p(T) \cdot T = 0$. Note that since $U \cdot T = 0$, we have that since $S_p(T) = -\nabla_T U$, $-S_p(T) \cdot T + U \cdot T' = 0$, so $S_p(T) \cdot T = U \cdot \kappa N = 0$. Since $\kappa \neq 0$ we know that U orthogonal to N.

Given these we then get that T, N are basis for the tangent plane, and B is the unit normal vector. Then the matrix of the shape operator with respect to $\{T, N\}$ basis is

$$\begin{pmatrix} S_p(T) \cdot T & S_p(T) \cdot N \\ S_p(N) \cdot T & S_p(N) \cdot N \end{pmatrix}$$

The Gaussian curvature is the determinant of the shape operator, which is $-(S_p(T) \cdot N)^2$, since we discovered that $S_p(T) = -B'$, so via Frenet Frames we have $S_p(T) = -B' = \tau N$, so $S_p(T) \cdot T = 0$, and $S_p(T) \cdot N = \tau N \cdot N = \tau$. Therefore $K = -\tau^2$. Note that this holds when $K < 0$.

3.3 T1

Let X be a Hausdorff space with two compact subsets A, B. Compact subspaces of Hausdorff spaces are closed, so we're asked to prove that X is a normal space. Consider the collection of open sets in B corresponding to a fixed $a \in A$, by which I mean choose $a \in A$, and then for each $b \in B$ there exists $U_{a,b}, V_{a,b}$ where $a \in U_{a,b}, b \in V_{a,b}$ and they're disjoint as X is Hausdorff. As we have $V_{a,b}$ for all $b \in B$ we have an open cover of B such that there is a finite subcover: $V_{a,b_1}, ..., V_{a,b_n}$. Each of these V_{a,b_i} are disjoint from the $U_{a,b}$ and still cover B. Take the intersection of the U_{a,b_i} corresponding to the finite subcover to get:

$$a \in U_a = \bigcap_{i=1}^{n} U_{a,b_i}, \quad B \subset V_B = \bigcup_{i=1}^{n} V_{a,b_i}$$

We've show that for each $a \in A$ and compact subset B there exists disjoint opens containing a and B. Do this for each $a \in A$ to get an open cover of A: $\{U_a\}$ for which there is a finite subcover $\{U_{a_i}\}$, and take the union of these open covers to get an open set $U = \cup_{i=1}^n U_{a_i}$ that contains A. By the above process each of the finite subcovers has a union of finitely many open sets containing B: V_{B_i}, take the intersection of this to get an open set $V = \cap_{i=1}^n V_{B_i}$ containing B.

3.4 T2

a)

We use Seifert van Kampen with the open sets $U = \{(x, t) : t \geq 1/3\}, V = \{(x, t) : t \leq 2/3\}$. Then I claim that these are contractible spaces. We do this via the homotopy $H((x, t), s) = (x, \text{linear map t } \mapsto 1)$. Via this at $s = 0$ we are at (x, t), and at $s = 1$ we get $(x, 1)$ which is a point for U. A similar homotopy for V sending $t \mapsto 0$ gives that V is contractible. So via SvK we get

$$\pi_1(X) \cong 1 *_{\pi_1(U \cap V)} 1 = 1$$

b)

Consider two points with the discrete topology. Then the suspension of this space will be S^1: Visually we attach vertical intervals to the two points, and get points at $t = 1$, then join the two points at the top and bottom to get a circle.

c)

We'll show $S(X)$ is path connected for any space, and then via part a we'll get $S(S(X))$ is contractible for free. Showing the space $S(X)$ is path connected can be done by considering $a = (x, t), b = (y, t) \in S(X)$. Then moving $t \to 0$ can be done since the interval is path connected, so we can slide along the interval to 0. Then a, b can be sent to $(x, 0), (y, 0)$, respectively. But this is the bottom point, and hence $(x, 0) = (y, 0)$ thus there is a path connecting a and b.

3.5 T3

We'll eventually use some path-lifting, and to do this we'll need the universal cover of X_k, but to do this we'll try to find the Fundamental Group of it. First we note that since $\mathbb{R}^3 \setminus \{0\} \simeq S^2$, then

$$\mathbb{R}^3 \setminus \{\text{k lines through 0}\} \simeq S^2 \setminus \{\text{2k points}\} \cong \mathbb{R}^2 \setminus \{\text{2k-1 points}\} \simeq \vee_{2k-1} S^1$$

Lets break the above down: The first homotopy equivalence is because we can expand the origin in our standard $\mathbb{R}^3$ minus a point to S^2 argument,and each line hits the sphere twice. The second homeomorphism, not just homotopy, is because one of the points, say WLOG the North pole, can be used in a sterographic projection argument to see that S^2 minus a point is homeomorphic to $\mathbb{R}^2$, then thus we have $\mathbb{R}^2$ minus $2k - 1$ points. Finally the last homotopy equivalence is that $\mathbb{R}^2$ minus n points is homotopy equivalent to a bouquet of circles. The fundamental group of the last space, via repeated SvK is the free group on $2k - 1$ generators.

The infinite tree graph E is a covering space for this, and via the hint it's contractible, implying it's the universal covering space. By the lifting theorem there is a lift of f that factors through the (contractible) universal cover, hence f is nullhomotopic.

3.6 T4

a)

We use Seifert van-Kampen. Take the open path connected sets to be $U = X - \{0\}$, the space minus the origin, and V to be a small open neighborhood of the missing point. Then U deformation retract to the boundary, which is just S^1, V is contractible, and $U \cap V$ is an annulus which deformation retracts to a circle. Thus by SvK we have

$$\pi_1(X) \cong \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) \cong \mathbb{Z} *_{\mathbb{Z}} 1$$

Where the presentation for the fundamental group of X is $\langle a \mid \iota_U(b) = \iota_V(b)\rangle$, where $i_U : \pi_1(U \cap V) \to \pi_1(U)$ induced by inclusion. $\iota_V = 1$, and under the inclusion map the generator b of $\pi_1(U \cap V)$ is sent 3 times around the circle, as rotation $2\pi/3$ 3 times give the identity so the fundamental group of X is

$$\pi_1(X) = \langle a \mid a^3\rangle \cong \mathbb{Z}_3$$

b)

Via the Classification of Compact Surfaces we know that every compact surface has fundamental group with no torsion or 2 torsion. Our space has fundamental group 3 torsion, and hence is not a surface.

3.7 T5

The Mobius band deformation retracts to S^1, and as cohomology classes are invariant under deformation retraction we compute the deRham Cohomology groups for S^1:

$H^0_{dR}(S^1) = \mathbb{R}$ as S^1 is connected. A generator is any continous function with zero derivative, so take $f(x) = 1$ for all $x \in M$.

$H^1_{dR}(S^1) = \text{Hom}\left(\frac{\pi_1(S^1)}{[\pi_1(S^1),\pi_1(S^1)]}, \mathbb{R}\right) \cong \text{Hom}(\mathbb{Z}, \mathbb{R}) \cong \mathbb{R}$. A generator for the 1st deRham group is $\omega = \frac{xdy - ydx}{x^2+y^2}$

S^1 is a 1-manifold so the higher deRham groups are zero, in particular, $H^2_{dR}(S^1) = 0$

4 January 2020

4.1 G2

Let $M \subset \mathbb{R}^3$ be a smooth compact connected surface with a unit normal vector field U that defines the Gauss map $G : M \to S^2$. If G is a local diffeomorphism we need to find all possible M. I claim that the only possible M is S^2.

If G is a local diffeomorphism, then by the Inverse Function Theorem, there are open neighborhoods V, W for which $p \in V, G(p) \in W$ such that the restriction of G to V is a diffeomorphism between V, W. Now as G is a local diffeomorphism, it's an open map, and having no kernel means G is injective. As such, $G(M)$ is open and nonempty, as G is injective. Since M is compact and connected, it's image under G is also compact and connected. As S^2 is Hausdorff, $G(M)$ is closed. S^2 is connected, and so $G(M) = S^2$ meaning that G is surjective. If $K \subset S^2$ is compact then K is closed, and thus $G^{-1}(K) \subset M$ is closed and therefore compact, hence G is proper. A surjective proper local diffeomorphism between Hausdorff spaces which the domain is compact and the codomain is connected is a covering map. Another way: A smooth map between connected manifolds which is proper and a surjective local diffeomorphism is a covering map.

All this to say that if G is a covering map, M is a covering space of S^2. But the only space that covers S^2 is S^2 itself, since S^2 is simply connected. Therefore $M \cong S^2$.

4.2 T1

4.3 T2

The Euler characteristic of T^2, K is 0, hence so is the Euler characteristic of their disjoint union. We get the above space by performing the following operation 4 times: Delete two discs from an existing surface, and glue the remainder along their boundaries. Deleting a disc is -1 to χ, identifying 3 vertices and edges has a net gain of 0. In total we delete 8 circles: 4 from the 4 holed torus, 4 from K, thus we have a net change of -8, thus our surface is non-orientable and hence $2 - g = -8 \implies g = 10$. The first homology is the abelianization of π_1 thus $\mathbb{Z}_2 \oplus \mathbb{Z}^9$

4.4 T3

Let Σ_g be the compact orientable surface of genus g. We use covering space actions to show this. Arraign Σ_4 to be a "fidget spinner": a central hole with 3 surrounding arms containing a single hole spaced by 120 degrees. Then $\mathbb{Z}_3$ acts on this space by rotating about the central hole. This act has no fixed points, and we can define a neighborhood around each point that is disjoint from the image of the neighborhood under the action. Hence this is a properly discontinuous group action (covering space action) (follows from the fact that we have a finite group acting on a Hausdorff space with no fixed points, thus it's properly discontinuous).

Because $\mathbb{Z}_3$ acts as a covering space action the quotient group $p : \Sigma_4 \to \Sigma_4/\mathbb{Z}_3$ is a normal covering space. This quotient space is a surface of genus 2, via cutting of the remaining arms under the action and gluing the two boundary circle together. For the covering map $p : \Sigma_4 \to \Sigma_4/\mathbb{Z}_3$, $p_*(\pi_1(\Sigma_4)) \subset \pi_1(\Sigma_2)$ is a subgroup isomorphic to $\pi_1(\Sigma_4)$ by injectivity, and

$$\mathbb{Z}_3 \cong \pi_1(\Sigma_4/\mathbb{Z}_3)/p_*(\pi_1(\Sigma_4)) \cong \pi_1(\Sigma_2)/\pi_1(\Sigma_4)$$

4.5 T4

A, B, C are all convex spaces hence contractible, D is not. So A, B, C are homotopy equivalent. But only A, C are homeomorphic, B has 3 connected components, but A, C have 4.

4.6 T5

$\mathbb{R}^3 \setminus S^1 \simeq S^2 \vee S^1$. Visualize $\mathbb{R}^3$ as a solid cube, then removing a circle from it still retains the inner part, we send the solid cube to a solid sphere, with no circle, all points outside the circle go to the sphere, all points inside the circle are sent to a line, and this space is $S^2 \vee S^1$. More concretely: Use SvK to find $\pi_1(M)$, let $U = \pi_1(\mathbb{R}^3 \setminus S^1)$ and $V = B_\epsilon(pt)$, then $U \cap V \simeq S^2$ and hence $\pi_1(M) \cong \pi_1(\mathbb{R}^3 \setminus S^1)$. Another way to visualize $\mathbb{R}^3 \setminus S^1$ is that this is a torus with a filled in center, all points outside the circle get sent to the surface of the torus, what's inside the circles radius fills in the hole. This core smushes to get a pinched torus. Thus $H^1 = \mathbb{R}$ Finally this space M is 2-dimesional so $H^3 = 0$ and it's connected so $H^1 = \mathbb{R}$

5 January 2019

6 August 2018

7 January 2018

7.1 G1

7.1.1 a

Use orthonormal expansion

$$T' = (T' \cdot T)T + (T' \cdot S)S + (T' \cdot U)U = (T' \cdot S)S + (T' \cdot U)U = aS + bU$$

7.1.2 b

Taking the norm of both sides $\kappa = \sqrt{a^2 + b^2}$, and geodesic curvature is the curvature along the perpendicular direction S, so $\kappa_g = a$

7.1.3 c

Gauss-Bonnet of a manifold says

$$\int_M K dM + \int_{\partial M} k_g ds = 2\pi\chi(M)$$

A sphere has Gaussian curvature $1/R$, and on the domain bounded by a curve, the region A is homeomorphic to a disc, so the above becomes

$$\frac{1}{R}\int_A dA + \int_{\partial A} k_g ds = 2\pi\chi(A)$$

This disc has Euler Characteristic 1 so we get

$$\int_{\partial A} k_g ds = 2\pi - \frac{1}{R} Area(A)$$

7.1.4 G3

Planes have a constant unit normal, hence $\nabla_{\alpha'} U = 0$ for the unit normal of α, as it shares the same unit normal for P, M, thus the shape operator is identically zero, and thus is asymptotic and trivially principal.

8 August 2015

8.1 G1

First we compute the derivative of the central curve:

$$\beta' = T(s) - \frac{\kappa'}{\kappa^2}N + (\frac{1}{\kappa})(-\kappa T + \tau B) + \frac{\tau}{\kappa}B = -\frac{\kappa'}{\kappa^2}N + \frac{\tau}{\kappa}B$$

If κ is constant then clearly β' is perpendicular to the osculating plane, since β' lives in the span of the binormal vector.

On the other hand if β' is perpendicular to the osculating plane, then $\beta' \cdot T = \beta' \cdot N = 0$, so this means that

$$\beta' \cdot N = -\frac{\kappa'}{\kappa^2} = 0$$

So $\kappa' = 0 \implies \kappa$ is constant

8.2 G3

The final answer should be

$$K = \frac{-f_{yy}f(x,y) + f_y^2 - f_{xx}f(x,y) + f_x^2}{(f(x,y))^4}$$

8.3 G5

Sketchy answer. Use the Gauss Bonnet Formula on the region R enclosed by the parallel γ:

$$\int\int_R K dM + \int_\gamma k_g ds + \sum \iota_j = 2\pi\chi(R)$$

Now, γ is a closed curve, so the sum of exterior angles is just 0. R is homeomorphic to a closed disc, so the Euler characteristic is 1. So far we have

$$\int\int_R K dM + \int_\gamma k_g ds = 2\pi$$

Next, the Gaussian Curvature of R is the Gaussian Curvature of the sphere for which R is a subset of, hence is r^{-2} where r is the radius of γ, however the radius

8.4 T2

We need that p is surjective and a set U in Y is open if and only if $p^{-1}(Y)$ is open in X. First, let $y \in Y$, then let $x = f(y)$. Then $p(x) = p(f(y)) = y$, so p is surjective. As p is continous, then if $U \subset Y$ is open, $p^{-1}(Y)$ is open. If $p^{-1}(Y) \subset X$ is open, then since f is continous, $f^{-1}(p^{-1})(U)$ is open in Y. But $f^{-1} \circ p^{-1} = (p \circ f)^{-1} = id_Y$, so $f^{-1}(p^{-1})(U) = U$

8.5 T3

Recall that via the Galois correspondence, covering spaces of T^2 are in bijective correspondence with subgroups of the fundamental group. However, $\pi_1(K) = \langle a, b : a^2 = b^2 \rangle$ is nonabelian, as $aba = b^{-1}$ is a relation, whereas $\pi_1(T^2) = \mathbb{Z} \times \mathbb{Z}$. Subgroups of abelian groups are abelian, and so this would require $\pi_1(K)$ to be abelian. As this cannot be true, K cannot cover the torus.

8.6 T5

We do some homotopies to these spaces to evenually compute π_1, if these spaces fail to share the same π_1 then they cannot hope to be homeomorphic.

$X_1 \simeq S^2 \vee S^2 \vee S^1$. The discs inside the torus can be contracted to points, giving a 'double pinced' torus. Expand out one of the pinches so that a string connectes these points, this space is $S^2 \vee S^2 \vee S^1$

$X_2 \simeq (S^1 \vee S^1) \times S^1$, akin to how the torus is $S^1 \times S^1$, the additional annulus introduces a wedge of circles. To see this (it's hard to visualize), start with a wedge of circles sitting 'upright', then at their connecting point rotate around the z-axis, to get a space which is X_2 once you contract the annulus to a circle.

Now these spaces don't have the same fund. gp so they cannot be homeomorphic.

Note: This should also be possible through Euler characteristic: Attaching two discs increasing your Euler characteristic by 2 (2 faces), and so $\chi(X_1) = 2$, but attaching an annulus to T^2

9 January 2015

9.1 G2

Let α be a unit speed curve on a surface M. Normally we have a Frenet frame $\{T, N, B\}$ associated to α, but this changes with the curve. Instead consider the orthonormal frame $\{T, U, U \times T\}$ where U is the unit normal of M. By definition $\kappa = ||\alpha''||$, so we'll use this.

By defintition $k_g = \langle \alpha'', U \times T \rangle$, and $k_n = \langle \alpha'', U \rangle$. Since $\{T, U, U \times T\}$ form an orthonormal frame we get that we can write α'' as a linear combination of these basis vectors by orthonormal expansion*:

$$\alpha'' = (\alpha'' \cdot T)T + (\alpha'' \cdot U)U + (\alpha'' \cdot U \times T)U \times T$$

$\alpha'' \cdot T = 0$, and so

$$\alpha'' = (\alpha'' \cdot U)U + (\alpha'' \cdot U \times T)U \times T = (k_n)U + (k_g)U \times T$$

Taking the norm gives

$$\kappa^2 = k_g^2 + k_n^2$$

(*) Theorem 1.5 in O'Neill states that if e_1, e_2, e_3 define a frame at a point, then for any tangent vector v

$$v = (v \cdot e_1)e_1 + (v \cdot e_2)e_2 + (v \cdot e_3)e_3$$

10 August 2011

10.1 T4

Assume for contradiction that there do exist f, g maps with the property that $g \circ f = id_X$. In this case by functionality we get that $(g \circ f)_* = g_* \circ f_* = id_*$. As f_* has a left inverse this means that it's injective. But if that's true it must mean $\pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z} \leq \mathbb{Z}^2$, which is impossible as the latter is abelian and the former is nonabelian.

10.2 T5

10.2.1 a

$X_1 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, and $X_2 \cong \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, which can be found via playing the standard game with V,E,F.

10.2.2 b

There is a standard theorem which says that for an n-sheeted covering space $p : E \to B$ between finite CW complexes (so for us any compact surface obtained by cutting and gluing), $\chi(E) = n\chi(B)$. So if X_1 were to cover X_2 we'd have that $-1 = n(-2)$ for a positive integer n, this is impossible so X_1 cannot cover X_2. If on the other hand X_2 covered X_1 we'd have $-2 = n(-1)$ so $n = 2$ solves this. Yet still we cannot have this, if $n = 2$ this means the number of sheets, and therefore the index of the image of $\pi_1(X_2)$ would be 2. Index 2 subgroups are normal so no non-regular covering exists.

11 January 2010

11.1 G1

$B(s) = \langle a\sin(s), a\cos(s), b\rangle$, taking a derivative we see that $-\tau N = \langle a\cos(s), -a\sin(s), 0\rangle$. Since $\tau > 0$ taking the norm returns the same value, but moreover tells us that $\tau = a$:

$$\tau = |\tau| = |-\tau N| = |\langle a\cos(s), -a\sin(s), 0\rangle| = a$$

Thus we get that $N = \langle -\cos(s), \sin(s), 0\rangle$, then we make take the cross product $N \times B = T$ to get $T = \langle -b\cos(s), -b\sin(s), a\rangle$, recalling that $T = \alpha'$ we integrate each component to get $\alpha = \langle b\sin(s) + c, b\cos(s) + c, as + c\rangle$ for a constant c.

11.2 G2

11.3 G3

If $x : D \to M$ is a coordinate patch then the Jacobian of x is the function such that $x^*dD = J_x dM$, so

$$J_x dD(v, w) = x^*dM(v, w) = dM(x_*v, x_*w) = (x_*v \times x_*w)U(x(p))$$

Now,

$$\begin{aligned}
J_x dD(v, w) &= (x_*v \times x_*w)U(x(p))du \wedge dv \\
&= (x_u \times x_w) \cdot (x_u \times x_w)/|x_u \times x_w| du \wedge dv \\
&= ((x_u \cdot x_u)(x_w \cdot x_w) - (x_u \cdot x_w)^2)/|x_u \times x_w| du \wedge dv \\
&= |x_u \times x_w|^2/|x_u \times x_w| du \wedge dv \\
&= |x_u \times x_w| du \wedge dv
\end{aligned}$$

We have an identity $|x_u \times x_w|^2 = (x_u \cdot x_u)(x_w \cdot x_w) - (x_u \cdot x_w)^2 = EG - F^2$, hence $|x_u \times x_w| = \sqrt{EG - F^2}$. Now we know that $dD = du \wedge dw$ so this gives $x^*dM = \sqrt{EG - F^2} du \wedge dw$

11.4 G4

a)

First we compute $f_{\epsilon,*}$, if α is a unit speed curve such that $\alpha(0) = p, \alpha'(0) = u$ then $f_{\epsilon,*}(u) = (f_\epsilon \circ \alpha)'(0) = u + \epsilon(U \circ \alpha)'(0) = u - \epsilon S_p(u)$. Let $f_\epsilon(\alpha) = \gamma$ and $f_\epsilon(\beta) = \delta$ for another unit speed curve β such that $\beta(0) = p$, and $\beta'(0) = v$ where u, v are orthogonal unit vectors. Then as we just saw $\gamma'(0) = u - \epsilon S_p(u)$, and $\delta'(0) = w - \epsilon S_p(w)$. If we look at the cross product we get the following:

$$\begin{aligned}
\gamma'(0) \times \delta'(0) &= (u \times v) - \epsilon S_p(u) \times v - \epsilon S_p(v) \times u + \epsilon S_p(u) \times \epsilon S_p(v) \\
&= (u \times v)(1 - 2\epsilon H + \epsilon^2 K)
\end{aligned}$$

Since the surface normals are nonzero we get that this is parallel to $u \times v$ hence the two tangent planes are parallel.

b)

$J_{f_\epsilon} dM(u,v) = f_\epsilon^* dM_\epsilon(u,v) = dM(f_{\epsilon,*}(u), f_{\epsilon,*}(v)) = (f_{\epsilon,*}(u) \times f_{\epsilon,*}(v))U(p) = (u \times v)(1 - 2\epsilon H + \epsilon^2 K)U(p)$

Now recall that $dM = U(u \times v)$, so $f_\epsilon^* dM_\epsilon(u,v) = (1 - 2\epsilon H + \epsilon^2 K)dM$

c)

The surface area of M_ϵ is gotten by the integral $\int_{f_\epsilon(M)} dM_\epsilon = \int_M f_\epsilon^* dM_\epsilon$, thus

$$\int_M f_\epsilon^* dM_\epsilon = \int (1 - 2\epsilon H + \epsilon^2 K)dM = \int dM - 2\epsilon \int H dM + \epsilon^2 \int K dM$$

Now $\int dM = A(M)$, the area of M, and $\int K dM = 2\pi\chi(M) = 2\pi(2 - 2g)$

11.5 G5

11.6 T1

11.7 T2

Lifting lemma + $\mathbb{Z}^2$ is free abelian and $\mathbb{Z}_2$ has torsion + factor through contractible domain.

11.8 T3

11.9 T4

11.10 T5

a

First we show that Y is neither a deformation retract nor a retract of X. We recall that all deformation retracts are retracts, but not necessarily the other way around, thus if we can show that Y is not a retract of X it cannot be a deformation retract.

12 January 2009

12.1 G1

12.2 T1

I think it's $\mathbb{Z}$

12.3 T4

Δ is a diagonal on the torus X, which is a closed loop on the surface, hence is homeomorphic to S^1. S^1 is a retract of the torus but not a deformation retract. The latter reason is because S^1 has a different fundamental group then T^2. But S^1 is a retract since we can just project every point 'up' to the top edge of the square.

12.4 T5

Let $X = \mathbb{R}^2 - \{S^1\} \cup \{(0,0)\}$, then this space is homotopy equivalent to an annulus which is homeomorphic to S^1 hence the deRham groups are those of S^1

13 August 2008

13.1 G2

$H^2 - K \neq 0$ means that the integral of these functions never vanishes, as such M is a nonwhere umblic surface. The integral of K must have a point of positive curvature as M is compact, and thus $2\pi\chi(M) \geq 0$ if it's strictly positive we're a sphere, which is a compact all umbilic surface, this is impossible so it's a torus.

13.2 G5

Nonsingular means nonwhere vanishing derivative. By the Inverse Function Theorem f is a local diffeomorphism, and therefore is an open map, since it sends open sets to open sets. $f(M)$ is therefore open, closed, and compact. As f is nonsingular we must have $f(M) = N$ as the only clopen sets of a connected set are the whole space and 0. Thus $f(M) = N$ is compact.

13.3 T2

Let A be the set of fixed points. Then the space $M = \#^5 T^2 \setminus A$, has no fixed points. Triangulate the surface M to that each of the k fixed points is a vertex, then in the lift of this triangulation we have $5(V-k)+k$ vertices, and 5 preimages of E, F. The projection map $p : M \setminus A \to \#^5T^2/\mathbb{Z}_2 \setminus A$ is a covering space so we can relate Euler Characteristic:

$$\chi(M) = 5(V-k) + k - 5E + 5F$$

$\chi(M) = 2 - 2(5) = -8$ so

$$-8 = 5V - 5k + k - 5E = 5F = -4k + 5(\chi(\#^5T^2/\mathbb{Z}_2))$$

Now, this is the same as $-8 + 4k = 5\chi(\#^5T^2/\mathbb{Z}_2)$, and since $k > 0, \chi(\#^5T^2/\mathbb{Z}_2) \leq 2$ we get upper and lower bounds of the left side $4(-2+k)$:

$$-4 \leq -8 + 4k \leq 10 \implies 4 \leq 4k \leq 18 \implies 1 \leq k \leq 4$$

Moreover we must have that $5| - 8 + 4k \implies 5|4(-2+k) \implies 5| - 2 + k$, but this is not possible unless $k = 2$

Such an example is

13.4 T3

The universal cover of a direct product of 3 circles is $\mathbb{R}^3$, this is a contracible domain, and as S^3 is simply connected, as such we may lift to a map which factors through a contracible domain, thus nullhomotopic.

13.5 T5

We show S^1 isn't a retract of none of these spaces, as such it cannot hope to be a deformation retract.

14 January 2008

14.1 G3

We use the parameterization $x(u,v) = (u, v, u^2 - v^2)$ to compute the shape operator

$$x_u = (1, 0, 2u) \quad x_v = (0, 1, -2v)$$

$$U = \frac{x_u \times x_v}{||x_u \times x_v||} = \left(\frac{-2u}{\sqrt{4u^2+4v^2+1}}, \frac{2v}{\sqrt{4u^2+4v^2+1}}, \frac{1}{\sqrt{4u^2+4v^2+1}} \right)$$

By the Weingarten equations in the basis of x_u, x_v the shape operator is of the form:

$$S(x_u) = -U_u = -\left(\frac{MF - LG}{EG - F^2} x_u + \frac{LF - ME}{EG - F^2} x_v \right)$$

$$S(x_v) = -U_v = -\left(\frac{NF - MG}{EG - F^2} x_u + \frac{MF - NE}{EG - F^2} x_v \right)$$

$$E = x_u \cdot x_u = 1 + 4u^2$$

$$F = x_v \cdot x_u = 4uv$$

$$G = x_v \cdot x_v = 1 + 4v^2$$

$$L = U \cdot x_{uu} = \frac{2}{\sqrt{4u^2+4v^2+1}}$$

$$N = U \cdot x_{uv} = \frac{-2}{\sqrt{4u^2+4v^2+1}}$$

$$M = U \cdot x_{vv} = 0$$

Thus

$$S(x_u) = -\left(\frac{\frac{-2(1+4v^2)}{\sqrt{4u^2+4v^2+1}}}{4u^2+4v^2+1} x_u + \frac{\frac{2(4uv)}{\sqrt{4u^2+4v^2+1}}}{4u^2+4v^2+1} x_v \right) = -\left(\frac{-2(1+4v^2)}{(4u^2+4v^2+1)^{3/2}} x_u + \frac{2(4uv)}{4u^2+4v^2+1^{3/2}} x_v \right)$$

$$S(x_v) = -\left(\frac{\frac{-2(4uv)}{\sqrt{4u^2+4v^2+1}}}{4u^2+4v^2+1} x_u + \frac{\frac{2(1+4u^2)}{\sqrt{4u^2+4v^2+1}}}{4u^2+4v^2+1} x_v \right) = -\left(\frac{-2(4uv)}{4u^2+4v^2+1^{3/2}} x_u + \frac{2(1+4u^2)}{4u^2+4v^2+1^{3/2}} x_v \right)$$

Thus

$$S = \begin{pmatrix} \frac{2(1+4v^2)}{(4u^2+4v^2+1)^{3/2}} & -\frac{2(4uv)}{4u^2+4v^2+1^{3/2}} \\ \frac{2(4uv)}{4u^2+4v^2+1^{3/2}} & -\frac{2(1+4u^2)}{4u^2+4v^2+1^{3/2}} \end{pmatrix}$$

Yielding that

$$K = \frac{-4}{(4u^2+4v^2+1)}$$

$$H = \frac{4(v^2-u^2)}{(4u^2+4v^2+1)^{3/2}}$$

14.2 T2

Contract the cylinder to be touching the sphere, this can be done via a straight line homotopy sending $[-3,3] \mapsto [-1,1]$ for x. Then the resulting surface is a sphere with a cylinder through the middle, which is homotopy equivalent to the sphere with a 'handle' attached to the outside, this is homeomorphic to the torus, and hence $\pi_1(Y) \cong \mathbb{Z}^2$

14.3 T4

We use homotopy equivalence to find these spaces. First we note that the arc connecting the north pole and south pole of X is closed contractible sub-complex of X, thus joining these together gives a space homotopy equivalent to $S^2 \vee S^1$. For Y we contract the disc to a point to get a 'pinched torus', which is homotopy equivalent to a crescent moon shape with a line connecting them. This is homotopy equivalent to $S^2 \vee S^1$, hence $X \simeq Y$. For Z contracting the disc yields an 'hourglass' shape that is homotopy equivalent to $S^2 \vee S^2$. So Z is not homotopy equivalent to X or Y.

14.4 T5

This is the helicoid. Note that if $z = 0$ we get a normal circle in the xy-plane. The larger z becomes, the larger the radius of the circle becomes in either direction. This space has fundamental group $\mathbb{Z}$ gotten by the generator wrapping around the central circle (we can collapse to the circle $x^2 + y^2 = 1$). As a result

$$H^1_{dR}(X) \cong \mathbb{R} \quad \omega = \frac{ydx - xdy}{x^2+y^2}$$

Is the 1st deRham group with a nontrivial generator. As X is a 2 dimensional surface any groups $H^n(X) = 0, n > 2$, and since X is noncompact $H^2(X) = 0$. Finally this is a connected space thus $H^0(X) = \mathbb{R}$ with any nonzero constant function as generator, $f(x) = 1$ works.

15 January 2006

15.1 G5

Use the exterior derivative to find $d\omega = 0$: Let $f = \frac{x}{x^2+y^2}\, dy, g = \frac{y}{x^2+y^2}\, dx$

$$\begin{aligned}
d\omega &= \frac{\partial f}{\partial x} \wedge\, dx - \frac{\partial g}{\partial y} \wedge\, dx \\
&= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}\, dy \wedge dx - \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\, dx \wedge dy \\
&= \frac{-(x^2+y^2)+2x^2-(x^2+y^2)+2y^2}{(x^2+y^2)^2}\, dx \wedge dy \\
&= \frac{-2(x^2+y^2)+2x^2+2y^2}{(x^2+y^2)^2}\, dx \wedge dy \\
&= 0
\end{aligned}$$

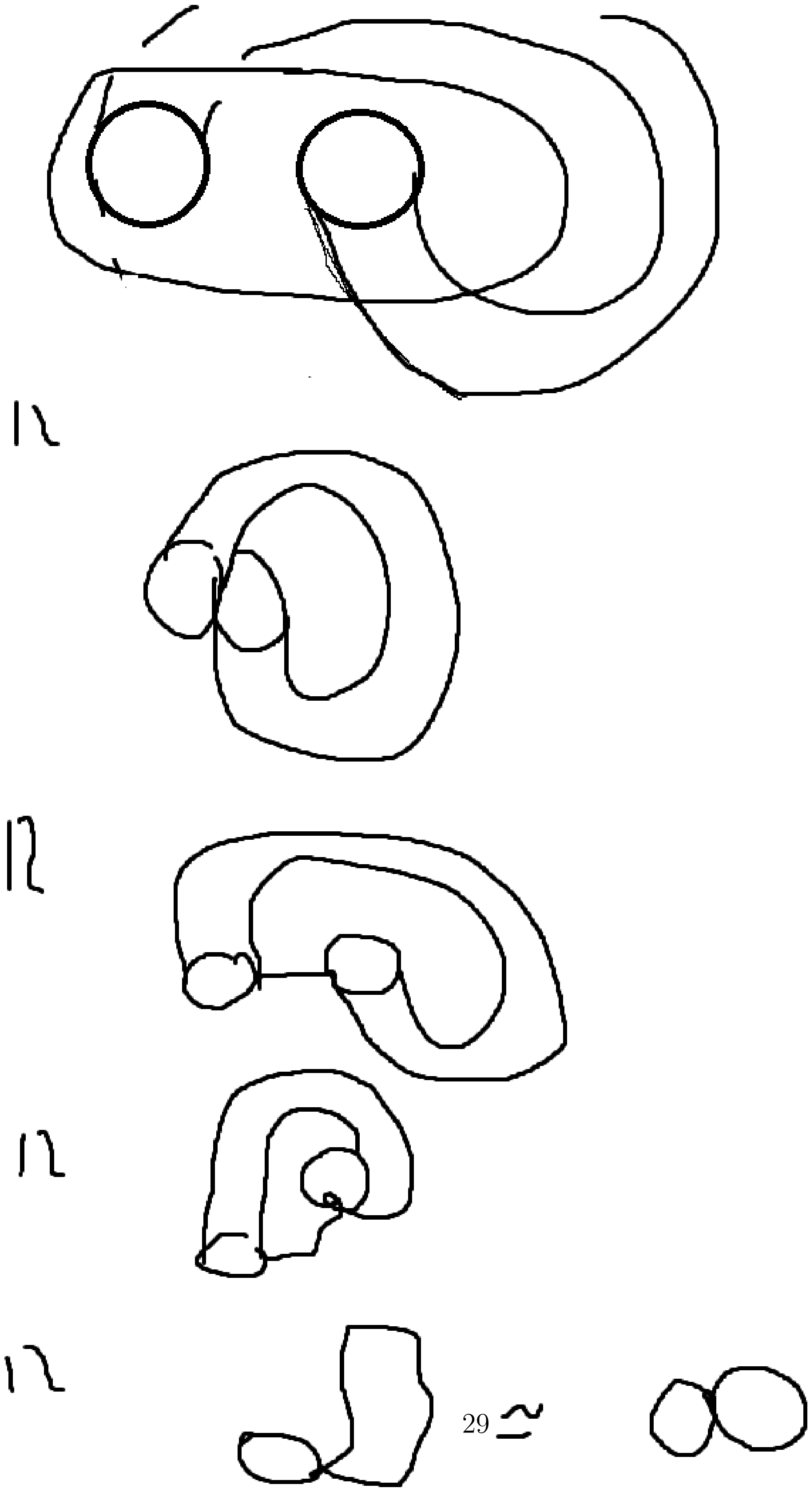
As for the second part, notice that the triangle with the prescribed vertices do not include the origin, as a result the curve C is contractible, and thus the integral is zero via Stoke's theorem:

$$0 = \int_{ABC} d\omega = \int_C \omega$$

15.2 T5

We use SvK. Let U be T^2 with an open arc around the two holed disk, this deformation retracts to just T^2 since this is just an open neighborhood around one of the generators of $\pi_1(T^2)$. Let V be the disk with the attached cylinder, and an open neighborhood around the disk, this just deformation retracts to the disk with the cylinder. $U \cap V \simeq S^1$. The fundamental group of $\pi_1(T^2) \cong \langle \alpha, \beta : \alpha\beta = \beta\alpha \rangle$, $\pi_1(U \cap V) \cong \langle \gamma \rangle$. For V we do some homotopy: Shrink the two holed disk down to $S^1 \vee S^1$ with a cylinder attached. Then expand a loop out from the middle of $S^1 \vee S^1$ to get something which look like a pair of glasses with a cylinder attached. The cylinder can be smushed down to give $S^1 \vee S^1$. (See picture below). Thus via SvK we get

$$\pi_1(Y) \cong \langle \alpha, \beta, \omega, \delta : \alpha\beta = \beta\alpha, \delta\omega = \alpha \rangle$$



16 August 2005

17 January 2005

(To fill in, sketch solutions)

17.1 G1

α unit speed means it's nonsingular, thus $\beta' = T + \epsilon(-\tau N)$ is nonsingular as $\epsilon > 0$ and $T \neq 0$. Use the formula

$$\kappa_\beta = \frac{||\beta' \times \beta''||}{||\beta'||^3}$$

17.2 G2

It's regular since $x_u \times x_v \neq 0$, thus can define a unit normal. Use the formula, and remember $S(x_u) = -U_u$

17.3 G3

17.4 G4

17.5 G5

17.6 T1

Probably want to use that the diagonal is closed iff Hausdorff

17.7 T2

See Lemma 75.2 in Munkres for a full solution. It requires path lifting.

17.8 T3

For contradiction assume that $h(x) \neq x$ for all $x \in M$, then the map $p : M \to M/\mathbb{Z}_p$ is a regular covering map, since we've asked for $\mathbb{Z}_p$ to act on the Hausdorff space M with no fixed points. As such we can define a triangulation of the surface $M/\mathbb{Z}_p$ to be such that pV, pE, pF, since we may cover the surface with a triangulation that has p preimages, and therefore we get

$$\chi(M) = p\chi(M/\mathbb{Z}_p)$$

As M is of genus 3 we know $\chi(M) = 2 - 2(3) = -4$ and we have that $p\chi(M/\mathbb{Z}_p) = p(V - E + F)$, thus $p| - 4$, but we know $p \geq 3$ is prime, hence this cannot happen. Contradiction.

17.9 T4

We need to do SvK 3 times. First let U be the left half of the 2-torus with an open collar of the right half, and V be the same with the right half. Then $U \cap V \simeq S^1$, it's an open cylinder, which smushes to a circle. Now we preform SvK on U, and the computation is the same for V.

For U we get a space which deformation retracts to a torus with the attached Mobius band around γ_1. Call this space X, and via abuse of notation let U be the torus with open nbhd of γ_1, V the Mobius band with open nbhd of its boundary, then $U \cap V$ is a circle. $\pi_1(U) = \langle \gamma_1, \alpha_1 : \gamma_1\alpha_1 = \alpha_1\gamma_1 \rangle$, $\pi_1(V) = \langle \xi : \emptyset \rangle$, $\pi_1(U \cap V) = \langle \delta : \emptyset \rangle$. By SvK we send δ to γ_1, since it's on the generator, and δ to ξ^2, since the boundary of the Mobius band $U \cap V$ goes twice around the inner circle, the generator of V. Thus by SvK we get

$$\pi_1(X) = \langle \gamma_1, \alpha_1 : \gamma_1\alpha_1 = \alpha_1\gamma_1 \rangle$$

17.10 T5

18 January 2004

18.1 T2

Let $f : S^n \to Y$ be a continuous closed surjective map. Let $y_1 \neq y_2$ be elements in Y. Then as f is surjective there exist $x_1, x_2 \in X$ such that $f(x_i) = y_i$ for $i = 1, 2$, and as S^n is Hausdorff it's singleton's are open, and under a closed map their images (y_i) are closed too. As such we may look at the pullback of these elements: $f^{-1}(y_i) \subset S^n$, which are closed subsets. Since S^n is compact Hausdorff it's normal, and therefore we may find open sets $f^{-1}(y_i) \subset U_i$. Consider now U_i^c, the complement of these open sets. Under f we get closed sets in Y: $f(U_i^c) \subset Y$, to get the sets in Y we want we take another complement: $y_i \in f(U_i^c)^c$

18.2 T3

Let Σ be the compact connected orientable surface of genus 2 and $h : \Sigma \to \Sigma$ a homeomorphism with order p. For contradiction assume there is no fixed point, then the action of h is properly discontinuous, and we get that $\Sigma/\mathbb{Z}_p$ is a surface, and $f : \Sigma \to \Sigma/\mathbb{Z}_p$, the quotient, is a covering map. Moreover each point in this quotient space has p-preimages. Triangulate the quotient space so that the V, E, F of the quotient has pV, pE, pF preimages. Then the Euler Characteristic of Σ is

$$\chi(\Sigma) = pV - pE + pF = p(V - E + F) = p\chi(\Sigma/\mathbb{Z}_p)$$

As Σ is the genus 2 surface we know $\chi(\Sigma) = 2 - 2(2) = -2$ which means that $p| - 2$, but p is an odd prime, yielding a contradiction.

18.3 T4

a)

A regular, or normal, covering is one for which between any two points $e_i, e_j \in p^{-1}(x_0)$ there is a deck transformation F such that $F(e_i) = e_j$. For this covering space E the deck transformation which shifts the line is one such deck transformation. As we may shift the line however far to the right or left and preserve the symmetry, this is a regular covering.

b)

As stated above the deck transformations of this group are those which shift the line left or right, thus giveing that $G \cong \mathbb{Z}$, geometrically the generator is that which shifts the points one step to the right: $F(e_i) = e_{i+1}$

c)

The subgroup is $\pi_(X, x_0)$, the whole group.

18.4 T5

If M is a smooth compact connected orientable surface with Euler Characteristic -4, then by the Classification of Compact surfaces, M is a 3 holed torus. Deleting a point from the surface gives a space X, which is the 'wire frame' of the surface, which under identification gives a wedge sum of 6 circles. Thus

$$H^0_{dR}(X) = \mathbb{R}$$

$$H^1_{dR}(X) = \bigoplus_{i=1}^{6} \mathbb{R} = \mathbb{R}^6$$

$$H^n_{dR}(X) = 0, \forall n \geq 2$$

19 January 2000

19.1 1

For part a just take enough derivatives. For part b, use the fact that the Frenet frame is a basis to write x in this basis with coefficients given by $x \cdot T$, etc. to get that part.

19.2 2

If M is a surface with a coordinate patch X in which there is a metric tensor

$$ds^2 = Edu^2 + Gdv^2$$

Then we can find a frame field of T_pM as follows: Let $E_1 = \frac{1}{\sqrt{E}}\frac{\partial}{\partial u}, E_2 = \frac{1}{\sqrt{G}}\frac{\partial}{\partial v}$. By construction this frame are multiples of the partials of X_u, X_v. This is orthonormal as we've divided by the norm of E, G and these are perpendicular to one another as well.

The dual frame is $\theta_1 = \sqrt{E}du$, $\theta_2 = \sqrt{G}dv$

19.3 6

The closure of a connected set is always connected: Assume $\overline{S} = A \cup B$, we aim to show A or B are empty.

19.4 7

Lemma. *If $\tilde{X}$, $\tilde{Y}$ are simply connected covering spaces of path/locally-path connected space X, Y respectively, then if $X \simeq Y$, $\tilde{X} \simeq \tilde{Y}$*

Proof. If $p : \tilde{X} \to X$, and $q : \tilde{Y} \to Y$ are the coverings then if $fg \simeq 1, gf \simeq 1$ are the respective homtopies of X, Y, we get a lift of $fp : \tilde{X} \to Y$ to $F : \tilde{X} \to \tilde{Y}$. This lift exists since $\tilde{X}$ is simply connected, so $fp_*(\pi_1(\tilde{X})) \subset q_*(\pi_1(\tilde{Y}))$, similarly we get a lift $G : \tilde{Y} \to \tilde{X}$. Now $GF : \tilde{X} \to \tilde{X}$ is homotopic to a deck transformation ϕ, so $\phi^{-1}GF \simeq id_{\tilde{X}}$ and likewise there is a deck transformation of $\tilde{Y}$ such that $FG\psi^{-1} \simeq \tilde{Y}$, thus we have a homotopy equivalence. $\square$

This fails for $n > 2$, but holds for $n = 2$. First let $n > 2$, then we know that $\mathbb{R}^n \setminus \{0\} \simeq S^{n-1}$ via the usual homotopy $x/||x||$. Then any covering map $f : \mathbb{R}^n \to S^{n-1}$ would imply that $\mathbb{R}^n$ is the univeral covering, but this cannot be true, as S^{n-1} is simply connected for $n > 2$ and therefore if such a covering exists it would mean that $\mathbb{R}^n \simeq S^{n-1}$ which is not true (the latter space is not contractible for example).

If $n = 2$ then we have a covering: $\mathbb{R}^2 \to S^1$ is a covering with the map....

20 January 1996

20.1 T6

Let U be $\Sigma_2 - pt$, V an open collar of the boundary of the Mobius band with the Mobius band, and $U \cap V \simeq S^1$. $\pi_1(U) \cong \langle a_1, b_1, a_2, b_2 \rangle$ (deleting point from the Σ_2 deform retracts to a wedge of circles), $\pi_1(V) \simeq \pi_1(M) \cong \langle \alpha \rangle$, and $\pi_1(U \cap V) \cong \langle \delta \rangle$. By SvK we get that $\pi_1(X) \cong \langle \alpha, a_1, b_1, a_2, b_2 : \alpha^2 = a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \rangle$

In order to identify the surface we look at what operations were preformed: We took Σ_2 and deleted a disk, effectively removing a face from the triangulation of the space. $\chi(\Sigma_2) = 2 - 2(2) = -2$, and deleting a disk reduces the Euler characteristic by 1: so now it's -3. Gluing on a Mobius strip identifies 3 vertices and 3 edges and thus has a net change of 0 to the Euler characteristic, hence we get that this new space has $\chi = -3$, it contains a Mobius band so it's nonorientable so we get that $2 - g = -3 \implies g = 5$ hence a connect sum of 5 projective planes.

20.2 T7

Removing the x-axis from 3-space deletes the origin, $\mathbb{R}^3 \setminus \{0\} \simeq S^2$, and the x-axis intersects S^2 at two points. WLOG take one of these to be the North Pole, then via Stereographic projection we get that this is homeomorphic to $R^2 - 0$, and thus $H^1 = Hom(\pi_1(S^1)^{ab}, \mathbb{R}) = \mathbb{R}$. The basis is given by $\omega = \frac{xdy - ydx}{x^2+y^2}$, which is closed by check, but not exact by integrating over $\alpha(t) = (\cos t, \sin t)$

21 January 1994